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Wavelet Analysis of One-Dimensional Cosmological Density Fluctuations

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Abstract

Wavelet analysis is proposed as a new tool for studying the large-scale structure formation of the universe. To reveal its usefulness, the wavelet decomposition of one-dimensional cosmological density fluctuations is performed. In contrast with the Fourier analysis, the wavelet analysis has advantage of its ability to keep the information for location of local density peaks in addition to that for their scales. The wavelet decomposition of evolving density fluctuations with various initial conditions is examined. By comparing the wavelet analysis with the usual Fourier analysis, we conclude that the wavelet analysis is promising as the data analysis method for the Sloan Digital Sky Survey and COBE.

1 Introduction

Recent large-scale sky survey (e.g. CfA) reveals the fertile large-scale structure of the universe [1]. It is natural to consider it as a consequence of gravitational instability from small-amplitude primordial fluctuations. The origin of the primordial fluctuations is usually ascribed to the quantum fluctuations around the Planck time which are stretched to a scale larger than the Hubble horizon during the inflationary stage and classicalize due to decoherence. Although the above picture is yet rather speculative, it encourages us to explain the large scale structure of the present universe from the physical point of view.

Based on the cosmological principle, the fairness of the observational data is assumed and the results are statistically interpreted. Then it is important to determine the statistical properties of the various cosmic fields that can be used to describe the matter distribution and motion in the expanding universe. For this purpose, the correlation functions are useful, indeed, two-point correlation function is well studied observationally which yields the fractal picture of the universe [1]. Other methods, such as the power spectrum, the topology of the iso-density surface [2] etc. are also used. As the inflation theory naturally predicts the random Gaussian initial fluctuations, the power spectrum analysis is useful as far as the density field stays in the linear regime. However, the nonlinear dynamics causes mode-mode coupling which produces non-Gaussianity generating reduced n-point correlation functions. In the terminology of the Fourier analysis, such nonlinear effect is described by the phase correlation. It was found in [3] that it is difficult to obtain information of nonlinear effect in this way.

Alternatively, one can works in real space, i.e., N-body simulation [4]. However, in real space, the information of the scale (wavenumber) is lacked. Thus, the Fourier spectrum analysis and the real space analysis is complementary. This situation is analogous to quantum mechanics. In quantum theory, the Wigner function which is defined on the phase space is known to be useful. Instead, in this paper, we propose to use the wavelet analysis [5] to characterize the density perturbation. In contrast with the Fourier analysis, localized basis functions are used in the wavelet analysis. Therefore, the information of the phase, or the position, is stored explicitly. In addition to the information of location, the wavelet analysis gives the scale information, which is similar to Wigner function in

quantum mechanics. Another advantage of the wavelet analysis should be stressed, that is, it can be performed with the data in certain compact region, in contrast the Fourier analysis needs the information of the whole domain.

To demonstrate its usefulness, we shall use one-dimensional cosmological model for which Zeldovich exact solution is known [6]. There are various ways to choose the basis functions in the wavelet analysis, however, as we concentrate on the non-linear effects it is sufficient to use an arbitrary basis, for which we choose the so-called Spline 4 wavelet. It is important to specify the useful statistics based on the wavelet analysis and calculate it from the physics and compare it with observation. To accomplish this task, thorough understanding of the non-linear dynamics based on the wavelet analysis is required. It is our present work that attack to this problem, as a first step in this direction. Our final goal is the application to the DSS project.

The plan of the paper is the following. In Sec.2, we review the wavelet analysis briefly. One-dimensional model is explained in Sec.3. The numerical results are presented in Sec.4. We have concluded with the summary and discussion of future problems in Sec.5.

2 Wavelet Analysis

Fourier transform method is a useful tool in data analysis since it enables us to decompose data into components with different scales. Many fundamental properties of physical systems have been described in terms of Fourier spectrum, that is, the amplitude of Fourier coefficients. However, since Fourier spectrum ignores the phase of each Fourier coefficient, it lacks information about positions of local events which are difficult to observe from the characteristics of the spectrum. The Fourier spectrum analysis therefore encounters difficulty in analyzing data which include different kinds of local structures.

Discrete wavelet analysis is invented to circumvent this difficulty. In the Fourier analysis, the basis functions of expansion are the familiar sines and cosines. In the wavelet analysis, the basis functions are somewhat more complicated localized functions, the so-called wavelet. Hence, it can detect both the location and the scale of the structure.

Let us review the wavelet analysis briefly. The reader is referred to the reference [5] for complete information. The essential ingredient of the wavelet analysis is the scaling

functions ϕ which satisfy the two-scale relation:

$$\phi(x) = \sum_{k\in Z} p_k \phi(2x-k) \ , \tag{1}$$

where $\{p_k\}$ is called two-scale sequence. The basis functions of Fourier analysis are $\exp ikx$ which can be regarded as dilation of $\exp ix$. In the wavelet analysis, the mother wavelet is localized, hence to cover the whole range it is necessary to perform the translation in addition to dilation. Discrete wavelet analysis uses 2^j dilation and the $k/2^j$ translation. Then, $\phi(2^j(x-k/2^j)) = \phi(2^jx-k)$. The index j denotes the level of the resolution and k represents the location. Let V_j be the space spanned by the scaling functions of level j, $\{\phi(2^jx-k)\}_{k\in Z}$. As the two-scale relation yields $\phi(2^jx) = \sum_k p_k\phi(2^{j+1}x-k)$, one sees $V_j \subset V_{j+1}$. Thus once the scaling functions are given, the hierarchical structure, called multiresolution analysis, is generated as

$$\cdots \subset V_{j-1} \subset V_j \subset V_{j+1} \subset \cdots \ . \tag{2}$$

Similarly, the mother wavelet is defined by the following two-scale relation:

$$\psi(x) = \sum_{k\in Z} q_k \phi(2x-k) \ . \tag{3}$$

Here the sequence $\{q_k\}$ is considered to define the mother wavelet. Let W_j be the space spanned by the wavelet functions of level j, $\{\psi(2^jx-k)\}_{k\in Z}$. Now, an arbitrary function $f_j \in V_j$ can be expressed as

$$f_j(x) = \sum_k c_k^{(j)} \phi(2^jx-k) \ , \tag{4}$$

where $c_k^{(j)}$ is the expansion coefficients. The multiresolution analysis suggests the direct sum decomposition,

$$V_j = V_{j-1} \bigoplus W_{j-1} \ . \tag{5}$$

The key fact is the uniqueness of this decomposition in the form

$$f_j = f_{j-1} + g_{j-1} \ , \tag{6}$$

where

$$g_j(x) = \sum_k d_k^{(j)} \psi(2^jx-k) \ . \tag{7}$$

Here, the expansion coefficients $d_k^{(j)}$ is the correspondent one to the Fourier coefficients. The major difference is the number of the index, that is, in the wavelet analysis there are two indices which indicate the location and the scale. The repeated decomposition gives

$$f_j = g_{j-1} + g_{j-2} + g_{j-3} + \cdots \ . \tag{8}$$

The above decomposition is the wavelet decomposition analysis.

If the data is given at 2^j points, one can construct interpolating function. This function is regarded as f_0. Then the above decomposition procedure gives

$$f_0 = g_{-1} + g_{-2} + \cdots + g_{-j} \ . \tag{9}$$

Of course there are many wavelets that have local support. We use the well-known spline wavelet of order 4 and Daubechies wavelet of order 4 as typical wavelets. We do not give their details; instead we show their plots in Fig. 1 and Fig. 2.

3 Model and Formalism

Throughout the present analysis, we adopt a matter dominated, spatially flat universe, i.e. the Einstein-deSitter model, for simplicity. Furthermore, we consider one-dimensional self-gravitating system by imposing plane symmetry. Let us consider only the situation in which the Newtonian approximation is valid. Then the basic equations are given by

$$\frac{\partial}{\partial t}\delta(x,t) + \frac{1}{a(t)}\frac{\partial}{\partial x}[v(x,t)(1+\delta(x,t))] = 0 \ , \tag{10}$$

$$\frac{\partial}{\partial t}v(x,t) + \frac{1}{a(t)}v(x,t)\frac{\partial}{\partial x}v(x,t) + \frac{\dot{a}}{a}v(x,t) + \frac{1}{a(t)}\frac{\partial}{\partial x}\phi(x,t) = 0 \ , \tag{11}$$

$$\frac{\partial^2}{\partial x^2}\phi = \frac{3}{2}(\frac{\dot{a}}{a})^2 a^2 \delta(x,t) \ , \tag{12}$$

where δ and v are the density perturbation and the peculiar velocity field, respectively and x is the comoving coordinate. Since we assume the flat universe, the scale factor $a(t)$ is proportional to $t^{2/3}$.

It is well known that there is an exact solution in a fully non-linear field, i.e. Zeldovich solution [6]. The basic idea underlying the Zeldovich solution is the transformation from Eulerian, x, to Lagrangian, q, space coordinates:

$$x = q + B(t)S(q) \ , \tag{13}$$

where $B(t)$ is a function of time to be determined later and $S(q)$ is an arbitrary function of q. Then the density fluctuation is explicitly given by

$$\delta(x,t) = -\frac{B(t)S'(q)}{1+B(t)S'(q)} \ , \tag{14}$$

where a prime denotes the derivative with respect to q. One can easily verify that the basic equations are solved automatically provided that $B(t)$ satisfies

$$\frac{d^2}{dt^2}B(t) + 2\frac{\dot{a}}{a}\frac{d}{dt}B(t) = \frac{3}{2}(\frac{\dot{a}}{a})^2 B(t) \ , \tag{15}$$

where a dot denotes a differentiation with respect to t.

The above equation is nothing but the same equation for the growth factor of pressureless matter in linear perturbation theory, and consists of a decaying and a growing solution. In what follows, we take the growing solution and use $B(t)$ as a time variable instead of t. In the linear stage, the density perturbation becomes

$$\delta(x,t) \sim -B(t)S'(x) \ , \tag{16}$$

hence the initial condition is determined by giving the arbitrary function $S(q)$. Also we set $B_{\rm init} = 1$. Let us consider the following form

$$S(q) = \sum_{k=1}^{k_c} \epsilon_n k^{n-1} \sin(kq + \alpha_k) \ , \tag{17}$$

where ϵ_n is independent of k, the power spectrum of the initial density fluctuations, $P(k)$, is essentially characterized by the spectral index $2n$ as $P(k) \propto k^{2n}$.

4 Numerical Results

We performed wavelet analysis for the Zeldovich solution of density perturbation in one-dimension. The numerical analysis was done for different initial power spectra given by $P(k) \propto k^{2n}$ and with the use of two mother wavelets. In this section, we show the numerical results for power spectra of $2n = 2, 0, -2$. Spline wavelet is mainly used in the analysis, while we show Daubechies wavelet analysis only for comparison.

We impose periodic boundary condition on a range $x \in [0, 2\pi]$ in one-dimension. The range is divided into evenly spaced $2^9 = 512$ small intervals. At each time t, the profile of

density fluctuation $\delta(t,x)$, given by the solution (14), is sampled at each intervals. Then the sampled data is transformed by wavelet transform using spline 4 wavelet. In this wavelet analysis, $\delta(t,x)$ can be decomposed into 9 levels, as is determined by the above discretization. The sampled data is at the same time transformed by Fourier transform so as to obtain power spectrum at each time. The time evolution is followed up to nonlinear stage, but well before the appearance of first orbit-crossing which occurs when the transformation (13) becomes singular so that our fluid approximation breaks down.

Initial profile of density contrast is given by the function $S(q)$ in (17), where α_k ($k = 1, \cdots, k_c$) are randomly given and we choose $k_c = 10$. The overall constant ϵ_n is chosen so that the initial amplitude of δ have maximum about 0.01. To be explicit, we tabulate them:

$2n$	ϵ_n
2	0.0004
0	0.002
-2	0.04

We first show the wavelet analysis for linear growth of density fluctuation, given by (13), so as to compare it with that for nonlinear evolution. Fig. 3 (a) depicts the time evolution of linear growth for $P(k) \propto k^2$, extrapolated to quasi-nonlinear stage. Fig. 3 (b)(c) show the wavelet analysis for the initial and late-time profile. (*Remark*: Our plot of the wavelet analysis, like Fig. 3 (b), is such that the relative magnitude of the amplitudes in 9 levels is meaningful. But the absolute magnitude is different from plot to plot. This remark applies to every plots of wavelet analysis below.) Corresponding to about 8 bumps of the density fluctuation we have signals at levels -5 and -6. In this linear growth of density fluctuation, we see no qualitative change of signals at each levels in the wavelet analysis (b) and (c). The magnitudes of signals in each levels simply increase in accordance with the linear growth of $\delta(t,x)$.

Fig. 4 (a)–(f) show the same case $P(k) \propto k^2$ for the identical initial condition as above. Fig. 4 (a) is the time evolution of density fluctuation up to full nonlinear stage. We wavelet-analyzed it for four stages — linear, quasi-linear, nonlinear and highly nonlinear stages and show the results in the sequence of Fig. 4 (c)–(f). In contrast with the linear case above, we can observe that there is transfer between levels. Indeed, the level $j = -4$ signal slightly appear from linear (c) to quasi-nonlinear stage (d). As the density fluctuation increases, higher levels become to have signals as is seen from (e) and (f). This "power-transfer

between levels" clearly represents the mode-mode coupling in quasi- and full-nonlinear growth.

Moreover, because density peak becomes more and more narrow due to gravitational effect, each peak makes a signal at the corresponding locus and in higher and higher levels during the time evolution. The wavelet analysis explicitly shows this situation in these figures. It is one of the advantages for the wavelet analysis that a kind of position-wavenumber representation is possible and yields enough information of the density peaks in real space. In the standard Fourier analysis, if one knows the power spectrum, one can in principle obtain such information from the phase correlation, but it is very inefficient to recover the real space structure in that way. In Fig. 4 (b), we show the power spectra for the same time evolution. This result is in agreement with the general tendency that the power-transfer occurs so that the power density tends to become equal among all modes [3].

We performed similar analyses for initial power spectra $P(k) \propto k^0$ and k^{-2}. The results are shown in Fig. 5 (a)–(f) and Fig. 6 (a)–(f) respectively. The "power-transfer between levels" is observed also in these cases.

In Fig. 5, additional feature can be found concerning a local structure of density peak. A bump present in the initial condition grows a strong peak in the nonlinear stage. (See the first peak in Fig. 5 (a).) As is found from the wavelet analyses for the late-time behavior in Fig. 5 (e) and (f), this strong density-peak yields a characteristic series of localized signals in the levels $j = -2, -3, -4$. This feature corresponds to a self-similar evolution of the density peak. In terms of the power-spectra in Fig. 5 (b), this corresponds to a power-law at high wavenumber appearing in the late-time spectrum there. This point has been studied in [6] [7], where it was shown that independently of initial conditions $|\delta_k|$ becomes to obey $k^{-1/3}$ power-law asymptotically. This universal behavior occurs prior but rather closely to the orbit-crossing singularity. However, this feature may be uninteresting in the viewpoint of large-scale distribution of galaxies, in which one is concerned with some "average" density field rather than the appearance of singularity.

In Fig. 6, while a similar feature appears more prominently in (e) and (f), we have still another interesting observation. For this initial condition, a rather large-scale void is formed where δ is negative. Since its wavelength is almost 2π, the most coarse-graining level -9 has the corresponding signal in the wavelet analyses (c) and (d). The signal

remains to be observed in spite of the presence of the strong density peak that was mentioned above. (See (e) and (f).) Wavelet analysis is also well adapted for the study of such global structures as well as the local ones.

The last result that we present is for comparison between the spline 4 wavelet analysis and that by Daubechies 4 wavelet. Fig. 7 (a)–(d) show the Daubechies wavelet analysis for the initial power spectrum $P(k) \propto k^2$ with the identical initial condition as the above Fig. 4. It can be seen from the result that signals in each levels are more distinctive than in the spline wavelet, so the Daubechies wavelet is sensitive in analyzing the local structure. The basic feature is not different between the two wavelet analyses.

5 Conclusion and Discussions

In order to understand the large scale structure of the universe, it is important to see what aspects we should look at. The power spectrum analysis is used so far. In the near future, the Sloan digital sky survey project (DSS) will begin and make a map of the whole universe. Hence, more elaborative tool of analysis is necessary to characterize the huge data. In this paper, we proposed the wavelet analysis as a new tool. In order to study its usefulness, we utilized the one-dimensional cosmological model and examined the non-linear dynamics from the point of view of the wavelet analysis. In the calculation of the dynamical evolution, we made use of the Zeldovich exact solution. First of all, the results of the linear evolution indicates no power-transfer between levels. In the full non-linear analysis, three type of the initial conditions are prepared. Due to non-linear interactions, higher level structures are generated in the location of the peaks. For comparison, we also presented the power spectrum evolution. We found that formation of local strong density-peak can be studied both from the real and "level" spaces in the wavelet analysis. This is based on its mathematical scheme that one can locally zoom-in and zoom-out a data. Furthermore, for completeness, we presented the analysis using different mother wavelet.

If one looks at the power spectrum, the initially prepared mode does not grow in the non-linear region as was studied previously (see [8] for example). On the other hand, in the real space, the density fluctuations will unboundedly grow in some spatially compact region due to nonlinear effect. This apparent contradiction is clearly a hasty consequence

of the Fourier analysis and reveals a disadvantage of the method. The underlying reason of this misleading argument is rather trivial, that is, the power-transfer accumulatively generates high frequency modes of small amplitude. Thus, in order to follow the system that evolves into the nonlinear stage, it is necessary to know the information about which modes are newly generated and how they form a structure in the real space at the same time. Clearly the Fourier analysis is not adequate for doing it. It is wavelet analysis that can detect the location (phase) of the newly generated structure. In the wavelet analysis, it is easy to see that accumulation of these structures is compensating the slowing down the growth of the original mode. Thus, it turned out that the wavelet analysis is very useful to understand the non-linear mode-mode interaction.

It should be stressed, however, that we regard the wavelet and Fourier analyses as complementary to each other. The power spectrum in Fourier analysis is an important tool that bridges theoretical prediction and the observational data. By using an orthogonal wavelet, such as Daubechies wavelet, it is possible to define a power spectrum in terms of the wavelet and to relate it with that in Fourier analysis. We may use the former to argue a kind of "local spectra" for positions of different density contrast. (Indeed, though in different field, recent study of turbulence in fluid mechanics found such a concept useful [9].) Since the wavelet analysis is entirely based on a similarity of the two-scale relation as was explained, it would be well adapted for the study of gravitational system where local and similar evolution is considered to be important. Statistical average of such local spectra would yield a new statistical description of the large-scale structure of the universe. Theoretical models could be analyzed in terms of the wavelet and be compared with such detailed statistical quantities. In a separate paper, we will report the investigation of these statistics.

As other future tasks, the extension to two- or three-dimensional space is planed. In addition to the DSS, it is interesting to apply the wavelet analysis to COBE data. Because the wavelet analysis does not require the whole sky data, the galactic plane does not cause any trouble in contrast to the spherical harmonics analysis. What is more important is how to compare such observational data obtained by the wavelet analysis with physical theories and models. Our strategy is the following; first, we will define the useful statistics such as the local power spectrum. It gives more information than the usual power spectrum. Next, the local power spectrum is measured from the observation. On the other

hand, we can calculate that quantity from the physical theory like as the inflation theory. Comparing both quantities will give powerful constraints on the theoretical parameters. Thus we believe that the wavelet analysis is quite promising and will open a new way for understanding the large scale structure of the universe.

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Figure Captions

Fig. 1 Spline 4 wavelet.

Fig. 2 Daubechies 4 wavelet.

Fig. 3 (a) Time evolution of linear growth of density fluctuation for initial power spectrum with $2n = 2$ at $B(t) = 10, 45, 74.2$.

Fig. 3 (b) Wavelet decomposition analysis for the initial profile at $B(t) = 1$.

Fig. 3 (c) Wavelet decomposition analysis for the late-time profile at $B(t) = 74.2$.

Fig. 4 (a) Time evolution of density fluctuation for initial power spectrum with $2n = 2$ at $B(t) = 10, 27.3, 45, 54.9$.

Fig. 4 (b) Power spectra at the corresponding times.

Fig. 4 (c) Wavelet decomposition analysis at $B(t) = 10$.

Fig. 4 (d) Wavelet decomposition analysis at $B(t) = 27.3$.

Fig. 4 (e) Wavelet decomposition analysis at $B(t) = 45$.

Fig. 4 (f) Wavelet decomposition analysis at $B(t) = 54.9$.

Fig. 5 (a) Time evolution of density fluctuation for initial power spectrum with $2n = 0$ at $B(t) = 10, 27.3, 60.7, 74.2$.

Fig. 5 (b) Power spectra at the corresponding times.

Fig. 5 (c) Wavelet decomposition analysis at $B(t) = 10$.

Fig. 5 (d) Wavelet decomposition analysis at $B(t) = 27.3$.

Fig. 5 (e) Wavelet decomposition analysis at $B(t) = 60.7$.

Fig. 5 (f) Wavelet decomposition analysis at $B(t) = 74.2$.

Fig. 6 (a) Time evolution of density fluctuation for initial power spectrum with $2n = -2$ at $B(t) = 1.5, 3.8, 8.3, 10$.

Fig. 6 (b) Power spectra at the corresponding times.

Fig. 6 (c) Wavelet decomposition analysis at $B(t) = 1.5$.

Fig. 6 (d) Wavelet decomposition analysis at $B(t) = 3.8$.

Fig. 6 (e) Wavelet decomposition analysis at $B(t) = 8.3$.

Fig. 6 (f) Wavelet decomposition analysis at $B(t) = 10$.

Fig. 7 (a) Daubechies analysis for the same time evolution as in Fig. 4. This is at $B(t) = 10$ corresponding to Fig. 4 (c).

Fig. 7 (b) The same as (a) and at $B(t) = 27.3$ corresponding to Fig. 4 (d).

Fig. 7 (c) The same as (a) and at $B(t) = 45$ corresponding to Fig. 4 (e).

Fig. 7 (d) The same as (a) and at $B(t) = 54.9$ corresponding to Fig .4 (f).